

Navier-Stokes Equation by Stochastic Variational Method

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We show for the first time that the stochastic variational method can naturally derive the Navier-Stokes equation starting from the action of ideal fluid. In the frame work of the stochastic variational method, the dynamical variables are extended to stochastic quantities. Then the effect of dissipation is realized as the direct consequence of the fluctuation-dissipation theorem. The present result reveals the potential availability of this approach to describe more general dissipative processes.

I. INTRODUCTION

In almost all branches in physics, the variational principle is one of the important guiding principles, and also serves as a powerful tool in practice [1]. In particular, this principle is indispensable method to deal with the symmetry property of a system, for example, the relativistic covariance. Once the symmetry of a system is expressed as the invariance of the action under the transformations of the symmetry in question, then the resultant equations of motion automatically satisfies this symmetry.

The usual variational principle, however, is not applicable when irreversible dynamics is present. Because dissipation involves energy exchange processes between macroscopic and microscopic motions, associated with the entropy production mechanism. These microscopic degrees of freedom are not included as the system degrees of freedom in the Lagrangian. Thus, dissipation is beyond the scope of the classical variational method [1]. To incorporate dissipative effects, we may introduce, for example, the Rayleigh dissipation function or some time-dependent external factor in actions, but it is not easy to specify uniquely them [2].

Three decades ago, a new variational approach called the stochastic variational method (SVM) was introduced [3, 4]. There, dynamical variables are extended to stochastic ones. In this way, dissipation is naturally induced by noise, without requiring additional functions. For example, it was shown that the Navier-Stokes (NS) equation for viscous fluids can be derived for an incompressible case [5, 6]. SVM is also used to derive the Schrödinger equation from an action of classical dynamics. In the latter case, SVM has been discussed in relation to Schrödinger and Nelson's formulation of quantum mechanics [7]. If the general criteria of the applicability of SVM is established, it will serve as a very powerful tool in practice. This is because, as we will see later, once we know the form of Lagrangian of the reversible dynamics, the effect of dissipation to this dynamics is introduced in a systematic manner. In general, the construction of Lagrangians is relatively easier than that of evolution equations, because the form of the Lagrangian is constrained from the symmetry principle.

In spite of these interesting concepts and features of the method, studies on its potential applicability to realistic problems have not been sufficiently developed. For example, even for NS equation, only the case of an incom-

pressible fluid has been studied. To claim the reliability of SVM, it is necessary to show explicitly that this can reproduce established dissipative equations.

In this letter, we show that the full NS equation, at least, can be derived in the framework of SVM. Our action is the non-relativistic version of that used for the variational formulation of relativistic hydrodynamics [8], but differs from those used in previous works of the classical variational formulation of hydrodynamics [1, 5].

II. ACTION FOR IDEAL FLUID

We first derive the Euler equation using the classical variational method. The same action used here is employed for the derivation of the NS equation in SVM.

Let us consider a fluid and divide it into small mass elements specified by their Lagrangian coordinate \mathbf{R} 's. The space position of the fluid element (the Euler coordinate) is denoted by

$$\mathbf{r} = \mathbf{r}(\mathbf{R}, t). \quad (1)$$

Let $\mathbf{v} = \mathbf{v}(\mathbf{R}, t)$ be the velocity field of the fluid element. When the time evolution is smooth, the time derivative of the Euler coordinate associated with the fluid element \mathbf{R} is given by this velocity field (see later discussion),

$$\left. \frac{\partial \mathbf{r}(\mathbf{R}, t)}{\partial t} \right|_{\mathbf{R}} = \mathbf{v}(\mathbf{R}, t). \quad (2)$$

Let U be the specific internal energy per unit mass of the fluid element. In general, in local thermal equilibrium, U depends on (\mathbf{R}, t) only through the specific entropy \hat{s} and the mass density ρ . With the energy density ε , we can write $U = \varepsilon/\rho$. The Lagrangian is then given by,

$$L = \int d^3\mathbf{R} \rho_0 \left(\frac{1}{2} \mathbf{v}^2(\mathbf{R}, t) - \frac{\varepsilon}{\rho} \right), \quad (3)$$

where the first and second terms represent, respectively, the kinetic energy and the "potential energy" associated with the fluid element. Here, ρ_0 is the mass density measured in the Lagrangian coordinate system and, by definition, does not depend on time. Note that the same Lagrangian can be expressed with the integral over the Euler coordinates as

$$L = \int d^3\mathbf{r} \left(\frac{1}{2} \rho \mathbf{v}^2(\mathbf{R}, t) - \varepsilon \right), \quad (4)$$

because, from Eq.(1), the two mass densities ρ and ρ_0 are related with the Jacobian $J = \det|\partial\mathbf{r}/\partial\mathbf{R}|$ through the coordinate transformation as $\rho = \rho_0/J$ [9].

Then the action is expressed as

$$I = \int_{t_a}^{t_b} dt \int d^3\mathbf{R} \rho_0 \left(\frac{1}{2} \mathbf{v}^2 - \frac{\varepsilon}{\rho} \right). \quad (5)$$

In the usual derivation of the Euler equation, the specific entropy \hat{s} should be kept constant for the variational procedure, representing that the fluid is ideal. Then we consider only the variation of \mathbf{r} , leading to

$$\left. \frac{\partial \mathbf{v}}{\partial t} \right|_{\mathbf{R}} + \frac{1}{\rho} \nabla_{\mathbf{r}} P = 0, \quad (6)$$

where P is pressure defined by the thermodynamic relation under the assumption of local thermal equilibrium,

$$P = - \frac{d}{d(1/\rho)} \left(\frac{\varepsilon(\rho, \hat{s})}{\rho} \right)_{\hat{s}}, \quad (7)$$

and the notation $\nabla_{\mathbf{r}}$ represents the gradient with respect to the Euler coordinates \mathbf{r} . In the following, the symbol ∇ is used only for the gradient with respect to \mathbf{r} , and hence the index is omitted. Equation (6) is the Euler equation, since

$$\partial_t \mathbf{v}(\mathbf{R}, t)|_{\mathbf{R}} = [\partial_t + \mathbf{v}(\mathbf{r}, t) \cdot \nabla] \mathbf{v}(\mathbf{r}, t). \quad (8)$$

In the above derivation, we used the following relations,

$$\sum_l \frac{\partial}{\partial \mathbf{R}^l} A^{il} = 0, \quad (9a)$$

$$\sum_k \frac{\partial \mathbf{r}^k}{\partial \mathbf{R}^i} A^{kj} = \sum_k \frac{\partial \mathbf{r}^i}{\partial \mathbf{R}^k} A^{jk} = J \delta^{ij}, \quad (9b)$$

$$\sum_l A^{il} \frac{\partial}{\partial \mathbf{R}^l} = J \nabla^i, \quad (9c)$$

where $A^{ij} = \partial J / \partial (\partial \mathbf{r}^i / \partial \mathbf{R}^j)$.

III. STOCHASTIC VARIATIONAL METHOD

Following the spirit of SVM, we start from the same Lagrangian (3) as the ideal case. In SVM [3, 5], however, we allow random fluctuations of \mathbf{r} in its time evolution due to noise, so that the time derivative of \mathbf{r} as Eq. (2) is not well-defined. The physical reason for the appearance of noise is the microscopic degrees of freedom which are coarse-grained in macroscopic scales. They act as the origin of fluctuations of the movement of fluid elements, leading to dissipation. Thus, the time evolution of \mathbf{r} is described by the following stochastic differential equation (SDE),

$$d\mathbf{r}(t) = \mathbf{u} dt + \mathbf{B} \star d\mathbf{W}(t). \quad (dt > 0), \quad (10)$$

where, in the right hand side, the first term \mathbf{u} is an unknown function of (\mathbf{r}, t) which is to be determined by the variational procedure, and the last term is the noise term given by a Wiener process $\mathbf{W}(t) = (W_x, W_y, W_z)$ satisfying

$$E[W_j(s) - W_j(u)] = 0, \quad (11a)$$

$$E[(W_j(s) - W_j(u))(W_k(s) - W_k(u))] = \delta^{jk} |s - u|, \quad (11b)$$

$$E[(W_j(s) - W_j(u))W_k(t)] = 0, \quad (11c)$$

where $j, k = x, y, z$, $-\infty < t \leq s, u < \infty$ and $E[\]$ denotes the expectation value of stochastic processes. The symbol \star is used to denote the Ito definition of a product for stochastic variables [10]. The coefficient \mathbf{B} is a vector composed of second rank tensors. For the sake of simplicity, we consider it as constant. We call Eq. (10) the forward SDE since it is defined only for $dt > 0$.

In SVM, the forward SDE is not sufficient to complete the formulation of stochastic variations. Because stochastic trajectories are not differentiable, the definition of velocity is not unique. There are two possible definitions for the velocity of a fluid element \mathbf{R} at t ,

$$\mathbf{v}_F \rightarrow \lim_{dt \rightarrow 0+} \frac{\mathbf{r}(\mathbf{R}, t + dt) - \mathbf{r}(\mathbf{R}, t)}{dt}, \quad (12a)$$

$$\mathbf{v}_B \rightarrow \lim_{dt \rightarrow 0-} \frac{\mathbf{r}(\mathbf{R}, t + dt) - \mathbf{r}(\mathbf{R}, t)}{dt}. \quad (12b)$$

When \mathbf{r} is continuous and smooth, the two definitions should coincide, $\mathbf{v}_F = \mathbf{v}_B$, as is the case of Eq. (2). However, stochastic \mathbf{r} is not smooth and we should distinguish the two evolutions defined by \mathbf{v}_F and \mathbf{v}_B .

The backward SDE which describes the time reversed process of Eq. (10) is given by,

$$d\mathbf{r}(t) = \tilde{\mathbf{u}} dt + \mathbf{B} \star d\tilde{\mathbf{W}}(t), \quad (dt < 0) \quad (13)$$

where the new drift term $\tilde{\mathbf{u}}$ should be related to \mathbf{u} ,

$$\mathbf{u}^i = \tilde{\mathbf{u}}^i + \sum_j 2\nu^{ij} \nabla^j \ln \rho. \quad (14)$$

This relation is obtained from the consistency condition of the two Fokker-Plank equations obtained from the two SDEs (10) and (13) [3, 5]. Here we introduced $\nu^{ij} = [\mathbf{B}\mathbf{B}^T]^{ij}/2$. The noise term $\tilde{\mathbf{W}}$ is again the Wiener process given by Eq. (11). There is no correlation between \mathbf{W} and $\tilde{\mathbf{W}}$.

It should be emphasized that the velocities \mathbf{u} and $\tilde{\mathbf{u}}$ are not parallel to the current of the mass density. From the Fokker-Plank equation obtained from Eq. (10), the mass density equation is uniquely given by

$$\partial_t \rho = - \sum_i \partial_i (\rho \mathbf{u}^i - \sum_j \nu^{ij} \partial_j \rho) = - \nabla \cdot (\rho \mathbf{v}_m). \quad (15)$$

Here \mathbf{v}_m is parallel to the mass current, and defined by,

$$\mathbf{v}_m^i = (\mathbf{u}^i + \tilde{\mathbf{u}}^i)/2 = \mathbf{u}^i - \sum_j \nu^{ij} \partial_j \ln \rho. \quad (16)$$

We call \mathbf{v}_m and \mathbf{u} the mass velocity and the diffusion velocity, respectively. For an incompressible fluid, all of three velocities, \mathbf{u} , $\tilde{\mathbf{u}}$ and \mathbf{v}_m coincide due to $\nabla\rho = 0$, showing the intrinsic difference between the compressible and incompressible cases.

Following Ref. [5], the action expressed with stochastic variables is then obtained from Eq. (3) by replacing \mathbf{v} with the mean forward derivative $D\mathbf{r}$,

$$I = \int_{t_a}^{t_b} dt \int d^3\mathbf{R} \rho_0 E \left[\frac{1}{2} (D\mathbf{r}(\mathbf{R}, t)) \cdot (D\mathbf{r}(\mathbf{R}, t)) - \frac{\varepsilon}{\rho} \right]. \quad (17)$$

Here the mean forward derivative $D\mathbf{r}$ is defined by

$$D\mathbf{r}(t) \equiv \lim_{h \rightarrow 0+} E \left[\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \middle| \mathcal{P}_t \right]. \quad (18)$$

Here, $E[F(t')|\mathcal{P}_t]$ denotes the conditional average of the time sequence $\{F(t'), t_a < t' < t_b\}$, taking the expectation values for $t' > t$, fixing $F(t')$ of $t' \leq t$ [7]. Thus $DF(t)$ is a stochastic variable. Note that the product of the mean forward derivatives is independent of the choice of the discretization scheme such as the Ito, Stratonovich-Fisk and Hänggi-Klimontovich schemes by the definition of the mean forward derivative [10].

The mean backward derivative is, similarly, defined as

$$\tilde{D}\mathbf{r}(t) \equiv \lim_{h \rightarrow 0+} E \left[\frac{\mathbf{r}(t) - \mathbf{r}(t-h)}{h} \middle| \mathcal{F}_t \right]. \quad (19)$$

where, $E[F(t')|\mathcal{F}_t]$ is the conditional average for the past sequence, fixing the future values.

There exist several ways to express the classical kinetic term in terms of $D\mathbf{r}$ and $\tilde{D}\mathbf{r}$. For example, $D\mathbf{r} \cdot \tilde{D}\mathbf{r}$ is used in Ref. [6]. The final results for the equation of motion depend on this choice. To obtain the NS equation, Eq. (17) is employed.

Similarly to the Euler equation, we consider variations for \mathbf{r} as follows,

$$\mathbf{r}_\lambda(\mathbf{R}, t) = \mathbf{r}(\mathbf{R}, t) + \lambda \mathbf{f}(\mathbf{r}, t), \quad (20)$$

where $\mathbf{f}(\mathbf{r}, t)$ is an arbitrary function with the boundary condition, $\mathbf{f}(\mathbf{r}, t)_{t=t_a} = \mathbf{f}(\mathbf{r}, t)_{t=t_b} = 0$ and λ is a small parameter. Then, for example, the variation of the mass density is calculated as

$$\delta_\lambda \rho = \rho_\lambda - \rho = -\lambda \frac{\rho}{J} \sum_{ij} \frac{\partial J}{\partial (\partial \mathbf{r}^i / \partial \mathbf{R}^j)} \frac{\partial \mathbf{f}^i}{\partial \mathbf{R}^j} + O(\lambda^2). \quad (21)$$

Keeping the terms up to first order in λ , $\delta_\lambda I$ is given as

$$\begin{aligned} \delta_\lambda I = & - \int_a^b dt d^3\mathbf{R} \sum_i \rho_0 \\ & \times E \left[\lambda \mathbf{f}^i \left(\tilde{D}D\mathbf{r}^i(\mathbf{R}, t) + \frac{1}{\rho} \frac{\partial}{\partial \mathbf{r}^i} P \right) + \frac{T}{m} \delta_\lambda \hat{s} \right], \quad (22) \end{aligned}$$

where $T = \partial m U / \partial \hat{s}$ and m is the mass of molecules. In this derivation, we used the stochastic partial integration formula, see Appendix. From Eq. (18) and the Ito formula [10], we obtain

$$\tilde{D}D\mathbf{r}^i = \tilde{D}\mathbf{u}^i = \partial_t \mathbf{u}^i + \sum_j \tilde{\mathbf{u}}^j \partial_j \mathbf{u}^i - \sum_{jk} \nu^{jk} \partial_j \partial_k \mathbf{u}^i. \quad (23)$$

Here the above expression corresponds for the general noise tensor \mathbf{B} . For the following discussion to derive the NS equation, we consider the case of $\mathbf{B} = \sqrt{2\nu} \mathbf{I}$, where \mathbf{I} is a unit matrix.

IV. ENTROPY VARIATION

Differently from the ideal case, \hat{s} should be treated as a functional of \mathbf{r} and we need to specify $\delta_\lambda \hat{s}$ in terms of Eq. (20). For this purpose, we employ a following simple model. Suppose that the fluid-dynamical time scale $\tau_{hyd} \equiv \rho/\dot{\rho}$ is much larger than that of microscopic degrees of freedom, τ_{mic} , inside a fluid element. Then local thermal equilibrium should be achieved, recovering the ideal fluid where \hat{s} is constant ($\delta_\lambda \hat{s} = 0$). Therefore, we expect that the entropy variation is expressed in powers of τ_{mic}/τ_{hyd} in such a way that the ideal fluid case is recovered in the vanishing limit of τ_{mic} . We then write $\delta_\lambda \hat{s} = \delta_\lambda (a_1 \tau_{mic} \dot{\rho}/\rho + a_2 (\tau_{mic} \dot{\rho}/\rho)^2 + \dots)$, where a_i 's are expansion coefficients. The lowest order truncation gives

$$\delta_\lambda \hat{s} = \delta_\lambda (g(\rho) \dot{\rho}), \quad (24)$$

where $g(\rho)$ is an arbitrary function of ρ . For the stochastic variation, $\dot{\rho}$ is interpreted as $(D + \tilde{D})\rho/2$. Note that $\delta_\lambda \hat{s}$ is the virtual change of \hat{s} associated with the variations and does not necessarily satisfy the thermodynamic principles such as $\delta_\lambda \hat{s} \geq 0$.

V. NAVIER-STOKES EQUATION

By substituting Eq. (24) into Eq. (22), the condition of $\delta_\lambda I = 0$ for the arbitrary function $\mathbf{f}(\mathbf{r}, t)$ leads to

$$\rho(\partial_t + \mathbf{v}_m \cdot \nabla) \mathbf{u}^i + \partial_i (P - \mu \nabla \cdot \mathbf{v}_m) - \sum_j \partial_j (\eta \partial_j \mathbf{u}^i) = 0. \quad (25)$$

Here we used that $\mu = -\rho^3 g(\rho)/m(\partial T/\partial \rho)_{\hat{s}}$ and $\eta = \nu \rho$. The contribution from $\delta_\lambda \hat{s}$ effectively changes pressure by $\mu \nabla \cdot \mathbf{v}_m$. The coefficient μ is known as the second coefficient of viscosity.

As was pointed out, the fluid velocity of the NS equation is not \mathbf{u} but \mathbf{v}_m . Eliminating \mathbf{u} using Eq. (16), Eq. (25) is finally re-expressed as

$$\begin{aligned} & \rho(\partial_t + \mathbf{v}_m \cdot \nabla) \mathbf{v}_m^i + \sum_j \partial_j [(P - \zeta \nabla \cdot \mathbf{v}_m) \delta^{ij} - \eta e_{ij}^m] \\ & - \sum_j \partial_j \left(\eta \partial_j \left(\frac{\eta}{\rho} \partial_i \ln \rho \right) \right) = 0, \quad (26) \end{aligned}$$

where

$$e_{ij}^m = \partial_j \mathbf{v}_m^i + \partial_i \mathbf{v}_m^j - \frac{2}{3}(\nabla \cdot \mathbf{v}_m) \delta_{ij}. \quad (27)$$

We thus identify η as the shear viscosity and $\zeta = \mu + 2\eta/3$ as the bulk viscosity. The last term is not only of second order for the magnitude of fluctuations $\nu = \eta/\rho$, but also of third order for the spatial derivative ∇ . In accordance with the approximation used in the NS equation, this term should be discarded as a higher order correction. In this sense, Eq. (26) is completely equivalent to the compressible NS equation.

VI. CONCLUDING REMARKS

In this letter, we showed for the first time that the stochastic variational method can be used to derive the full Navier-Stokes equation. The basic ingredients of this approach are the classical action of the ideal fluid and the stochastic motions of fluid elements which are induced by the white noise.

The present derivation shows clearly that there are two different physical origins for the shear and bulk viscosities. The shear stress tensor and a part of the bulk viscous pressure are obtained through the stochastic motions of fluid elements, while the entropy variation in the potential term affects only the bulk viscous pressure (the second coefficient of viscosity). It is easy to see that the entropy variation due to the shear stress tensor e_{ij}^m gives only higher order corrections. Thus, the shear viscosity in the NS equation is not altered by such a modification.

So far, we have emphasized the mathematical aspects of SVM and did not discuss its physical background. To see it, note that the noise introduced in Eq. (10) is related directly to the transport coefficients. For example, we can show that η satisfies the Einstein relation,

$$\eta = \frac{\rho}{3} \int_0^\infty dt E[\delta \hat{\mathbf{v}}(t) \cdot \delta \hat{\mathbf{v}}(0)], \quad (28)$$

where $\delta \hat{\mathbf{v}} = d\mathbf{r}/dt - \mathbf{u} = \sqrt{2\nu} d\mathbf{W}/dt$. This is nothing but the realization of the fluctuation-dissipation theorem and it appears as a natural consequence of SVM. We thus conclude that SVM possesses not only the well-defined mathematical structure but also a reasonable mechanism of dissipation.

Above results show that the SVM approach is considered as a promising framework, and can be extended to more general dissipative phenomena which are not in the scope of the NS equation. In fact, generalization of a diffusion equation is done in SVM [11]. Another possible example can be found in the soft matter physics. One

formulation of such dissipative equations is based on Onsager's variational method [12] so that the comparison of SVM to Onsager's method will clarify the physics of variational approaches for the dissipative phenomena.

As shown, the present result of SVM specifies the form of the higher order correction to the NS equation. If SVM is a reliable approach, this higher order term neglected in the NS equation should be considered seriously. The structure of Eq.(25) reminds that of the generalized hydrodynamics proposed by Brenner [13, 14], but in the SVM approach, the difference of the two velocity fields appears as the higher order correction to the NS equation. One example of the importance of higher order terms obtained by SVM plays in fact a crucial role in diffusion processes. The application of SVM to a diffusion process leads to a generalized form of the diffusion equation [15] which contains memory effects. The existence of the higher order term guarantees Fick's law [11].

As mentioned above, in spite of very attractive features, the general availability of SVM is not yet established. This question should be investigated by the applications of SVM to field theoretical systems, relativistic systems and complex fluids.

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Appendix A: stochastic partial integration formula

The time variable is discretized as

$$t_j = a + j \frac{b-a}{n}, \quad j = 0, 1, 2, \dots, n. \quad (A1)$$

Then we can show

$$\begin{aligned} & \int_a^b dt E[\{D\mathbf{X}(\mathbf{t})\} \cdot \mathbf{Y}(\mathbf{t}) + \mathbf{X}(\mathbf{t}) \cdot \tilde{\mathbf{D}}\mathbf{Y}(\mathbf{t})] \\ &= \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} E \left[(\mathbf{X}_{j+1} - \mathbf{X}_j) \frac{\mathbf{Y}_{j+1} + \mathbf{Y}_j}{2} \right] \frac{b-a}{n} \\ &+ \lim_{n \rightarrow \infty} \sum_{j=1}^n E \left[\frac{\mathbf{X}_j + \mathbf{X}_{j-1}}{2} (\mathbf{Y}_j - \mathbf{Y}_{j-1}) \right] \frac{b-a}{n} \\ &= \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} E[\mathbf{X}_{j+1} \mathbf{Y}_j - \mathbf{X}_j \mathbf{Y}_{j-1}] \\ &= E[\mathbf{X}(b) \mathbf{Y}(b) - \mathbf{X}(a) \mathbf{Y}(a)]. \end{aligned} \quad (A2)$$

This is called the stochastic partial integration formula [16].

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